8 Problems with partial-differential equations

In this topic, we will describe and look at finding approximations to solutions to

1. Laplace's equation,
2. the heat-conduction/diffusion equation, and
3. the wave equation.

In each case, we will use the finite-difference method.

Background

For a function of two variables, the first and second partial derivatives may be approximated by

$$\frac{\partial}{\partial x}u(x,y) = \frac{u(x+h,y)-u(x-h,y)}{2h} + \frac{1}{6}\frac{\partial^3}{\partial x^3}u(\xi,y)h^2$$
$$\frac{\partial}{\partial y}u(x,y) = \frac{u(x,y+h)-u(x,y-h)}{2h} + \frac{1}{6}\frac{\partial^3}{\partial y^3}u(x,\upsilon)h^2$$
$$\frac{\partial^2}{\partial x^2}u(x,y) = \frac{u(x+h,y)-2u(x,y)+u(x-h,y)}{h^2} + \frac{1}{12}\frac{\partial^4}{\partial x^4}u(\xi,y)h^2$$
$$\frac{\partial^2}{\partial y^2}u(x,y) = \frac{u(x,y+h)-2u(x,y)+u(x,y-h)}{h^2} + \frac{1}{12}\frac{\partial^4}{\partial y^4}u(x,\upsilon)h^2$$

The gradient is defined as a vector of the partial derivatives of a function:

$$\nabla u(x) = \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x}u(x) \end{pmatrix}$$
$$\nabla u(x,y) = \begin{pmatrix} \frac{\partial}{\partial x}u(x,y) \\ \frac{\partial}{\partial y}u(x,y) \end{pmatrix}$$
$$\nabla u(x,y,z) = \begin{pmatrix} \frac{\partial}{\partial x}u(x,y,z) \\ \frac{\partial}{\partial y}u(x,y,z) \\ \frac{\partial}{\partial z}u(x,y,z) \end{pmatrix}$$

The Laplacian is defined as the inner product of the gradient operator with itself, so

$$\nabla^2 u(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} u(x)$$
$$\nabla^2 u(x,y) = \frac{\partial^2}{\partial x^2} u(x,y) + \frac{\partial^2}{\partial y^2} u(x,y)$$
$$\nabla^2 u(x,y,z) = \frac{\partial^2}{\partial x^2} u(x,y,z) + \frac{\partial^2}{\partial y^2} u(x,y,z) + \frac{\partial^2}{\partial z^2} u(x,y,z)$$

Acknowledgments